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# The energy distribution resulting from an impact on a floating body

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The initial stage of the water flow caused by an impact on a floating body is considered. The vertical velocity of the body is prescribed and kept constant after a short acceleration stage. The present study demonstrates that impact on a floating and non-flared body gives acoustic effects that are localized in time behind the front of the compression wave generated at the moment of impact and are of major significance for explaining the energy distribution throughout the water, but their contribution to the flow pattern near the body decays with time. We analyse the dependence on the body acceleration of both the water flow and the energy distribution – temporal and spatial. Calculations are performed for a half-submerged sphere within the framework of the acoustic approximation. It is shown that the pressure impulse and the total impulse of the flow are independent of the history of the body motion and are readily found from pressure-impulse theory. On the other hand, the work done to oppose the pressure force, the internal energy of the water and its kinetic energy are essentially dependent on details of the body motion during the acceleration stage. The main parameter is the ratio of the time scale for the acoustic effects and the duration of the acceleration stage. When this parameter is small the work done to accelerate the body is minimal and is spent mostly on the kinetic energy of the flow. When the sphere is impulsively started to a constant velocity (the parameter is infinitely large), the work takes its maximum value: Longhorn (1952) discovered that half of this work goes to the kinetic energy of the flow near the body and the other half is taken away with the compression wave. However, the work required to accelerate the body decreases rapidly as the duration of the acceleration stage increases. The optimal acceleration of the sphere, which minimizes the acoustic energy, is determined for a given duration of the acceleration stage. Roughly speaking, the optimal acceleration is a combination of both sudden changes of the sphere velocity and uniform acceleration.

If only the initial velocity of the body is prescribed and it then moves freely under the influence of the pressure, the fraction of the energy lost in acoustic waves depends only on the ratio of the body's mass to the mass of water displaced by the hemisphere.

## 1. Introduction

Unsteady flow caused by a sudden vertical motion of a floating body is considered. The impact of rigid bodies on water and the impact of water on rigid bodies are often described within the ideal incompressible liquid model by using the pressure impulse concept. That is, the change in water motion due to impact is supposed to take place over such a short time scale that the convective nonlinear terms and the viscous terms in the equations of motion are taken as negligible compared with the direct acceleration term and the pressure gradient. This results in the concept of pressure impulse, which is a time integral of pressure through the impact. This impulsive model assumes an inelastic impact, and a consequent loss of energy from the flow. It is this 'lost' energy we consider here. For the sudden motion of a rigid body in an unbounded fluid the lost energy is carried away by acoustic waves, but when there is a bounding free surface energy may also be lost to thin jets and other small-scale motions. The distribution of lost energy in the modelling of a violent impact with an incompressible fluid bounded by a free surface is still an open question. It is of importance in the interpretation of experimental results of such flows and in numerical simulations.

### 1.1. *'Lost' energy in unbounded flows*

Here we consider the special problem of impact on a half-submerged sphere; the mathematical analysis for linearized free-surface boundary conditions is then identical to that of sudden motion of a sphere from rest in unbounded acoustic medium. This latter problem has been studied since Kirchhoff (1876) analysed the unsteady given motion of a rigid sphere in an acoustic medium. Love (1905, see Lamb 1932) studied a more general problem, where the sphere motion is unknown in advance and has to be found together with the fluid flow. The analysis given by Love and his main results are reproduced in the textbook by Lamb (1932). Later Taylor (1942) simplified Love's solution and applied it to the free motion of a sphere in a compressible fluid. The sphere is subjected to a sudden initial impulse and there is no restoring force apart from that provided by the compressibility of the fluid. Taylor showed that the resilience of the fluid is not capable of reversing the direction of motion of the body. He also derived the energy distribution after impact on a sphere. We discuss these results in § 7. The Kirchhoff problem of unsteady motion of a rigid sphere in an unbounded acoustic fluid is analysed in Longhorn (1952), where a formula is derived for the work required to accelerate a sphere up to a given velocity. The formula is quite complicated and requires double integration with respect to time. Longhorn studied two cases: impulsive start and uniform acceleration of the sphere. He found that the work required to start a sphere impulsively is twice the amount needed if the sphere is started gradually. The additional work goes to the energy radiated as sound. Ffowcs Williams & Lovely (1977) considered the case of impulsive start of a sphere also and found an equipartitioning of the energy between the kinetic energy of the fluid surrounding the sphere and the radiated acoustic energy.

Direct nonlinear simulations of the compressible water flow generated by a circular cylinder accelerating from rest were performed by Brentner (1990, 1993). Analysing the temporal and spatial characteristics of the numerical solution, he distinguished carefully the propagating acoustic energy, the convective energy associated with the entropy change in the fluid, and the energy following the body. He studied the energy distribution both in space and in time, as well as the global energy balance for a Mach number of 0.4. It was discovered, in particular, that the energy in the radiated acoustic wave is 'nearly equally divided between kinetic and potential energy components'.

There are many other relevant papers on the motion of a sphere in a compressible fluid; for more details the PhD thesis by Brentner (1990) and the book by Gorshkov & Tarlakovskii (1990) are recommended.

### 1.2. *'Lost' energy in flows with a free surface*

The distinguishing feature of the inclusion of a free surface is that energy may also be lost to smaller-scale incompressible motions. It is well known that after impact only a part of the original energy remains as kinetic energy of the visible flow, e.g.

see Logvinovich & Yakimov (1973), and Rogers & Szymczak (1997). On the other hand Cooker & Peregrine (1995) suggest that the lost energy of steep wave impact on a vertical wall goes to the small-scale motion, such as narrow jets, in a manner which probably strongly depends on the presence or absence of trapped air in the impact region. This mode of 'losing' energy is given strong support by comparisons with incompressible, irrotational computations for 'flip-through' described in Cooker & Peregrine (1990, 1992). Trapped air can lead to a more elastic impact and hence reduce the lost energy (Wood & Peregrine 2000). However, Korobkin's (1997) overview of the impact of a blunt body on a free surface clearly shows that both compressible and free-surface effects need to be considered.

In the field of water impact (slamming, breaking wave impact and so on with the impact velocities up to several metres per second) the model of an ideal and incompressible liquid is traditionally used. It is commonly believed that the relief of high impact pressures is mostly due to free-surface deformations, which can be correctly described within the incompressible liquid model, but not due to compression waves propagating away from the impacting body. This belief is based on two reasons.

(i) In the case of blunt-body impact onto a water free surface it is found that half the work done to move the body into the water at a constant velocity goes to the kinetic energy of the main flow and the other half is taken away with spray jets, which are very thin at the initial stage of the impact but with velocities far exceeding the impact velocity. This distribution of the energy from the impact of blunt bodies was first used by Logvinovich (1969) to determine the shape of the spray jets formed under wedge impact onto an incompressible free surface. Later, Howison, Ockenden & Wilson (1991) derived the equations which govern the water flow inside the jets caused by impact of a blunt wedge, and calculated the jet shape. The results obtained confirm Logvinovich's conclusion based on simple physical arguments. The problem of blunt-wedge impact is self-similar, which is why the velocities of the water particles inside the spray jets are very high but finite within the incompressible liquid model.

However, in the case of the impact of a body that is blunt and smooth, the incompressible liquid model predicts unbounded velocities close to the jet tips. In order to determine the flow within the spray jets in this case, Korobkin (1992, 1994*a*, *b*, 1997) developed an acoustic theory. The kinetic energies of spray jets generated by the impact of a smooth blunt body, which initially touches the free surface of a weakly compressible liquid at a single point, were evaluated for both plane and axisymmetric problems (Korobkin 1994*a*). Acoustic impact theory shows that for large times, when the compression waves are already far from the impact region, the kinetic energy of the spray jets (spray sheet in the axisymmetric problem) approaches the value of the kinetic energy of the main flow and their sum is equal to the work done to move the body into the water at a given constant velocity. Molin, Cointe & Fontaine (1996) obtained the same result but within the incompressible liquid model. They observed that the kinetic energy of incompressible spray jets is finite and equal to the energy of the flow in the main region even when the water velocity is unbounded at the jet tips.

These results demonstrate that the acoustic energy taken away from the body in compression waves is negligibly small compared to both the kinetic energy of incompressible flow near the body and the kinetic energy of spray jets. Therefore, the acoustic effects give a small contribution to the total energy of the water at the end of the acoustic stage in the problem of blunt-body impact. There are similarities to the initial motion of a wavemaker, see King & Needham (1994)

(ii) The second reason is connected with the success of pressure-impulse theory in applications. This theory makes it possible to determine the flow field just after the impact. The theory has been applied to the problems of ship hydrodynamics, impulsive motion of a body placed in an incompressible liquid and wave impact (Cooker & Peregrine 1992). The results obtained with the help of this theory are in good agreement with measured data, which suggests that for low impact velocities the water can be considered as incompressible. On the other hand, the kinetic energy of the flow after the impact and the work done to oppose the pressure force calculated within the pressure-impulse approach do not correspond to each other. It is expected that a correct description of nonlinear free-surface motion may properly explain where the 'lost' energy goes. It is found to be due to the rapid motion of the free surface close to the intersection line, which is not fully taken into account within the pressure-impulse theory (Cooker 1995).

In general, to provide a fully correct description of the water impact, analysis of the nonlinear flow of compressible liquid with free surface at the initial stage of body motion is required. This is too complicated for theoretical analysis at present, which is why we study different approximate models of the impact stage using asymptotic methods and making assumptions about both the flow pattern and the free-surface deformation. It is expected that the solution of the problem of lost energy strongly depends on the flow geometry and the impact conditions. In order to be specific, only the problem of sudden downwards motion of a floating body with vertical tangent at the waterline is considered in the present paper.

### 1.3. *The hemispherical problem*

We expect that the free surface has less influence if the jets, which are formed at the intersection of a body surface and the water surface, are weak. Although the concept of 'weak' and 'strong' jets is still not clear, we can say that in the case of blunt-body impact onto a water surface the spray jets are strong, with their energies being comparable with the energy of the main flow, and in the case of downward impact on a floating half-submerged sphere the jets are expected to be weak, perhaps with negligible energy, since the waterline motion of the rigid body is tangential to its surface. This and the existence of simple analytical solutions motivates the present study.

The general formulation of the problem is as follows. Initially, liquid at rest occupies the lower half-space and a body is floating at rest on the liquid surface with its submerged portion being hemispherical of radius $R$. The liquid is assumed to be inviscid and compressible, and the body is rigid. At some instant of time, which is taken as the initial moment, $t' = 0$, the body suffers an impact and starts to move down, and after a brief acceleration stage attains a constant velocity $V$. The problem is to determine both the flow and the pressure distribution immediately after the acceleration stage, to evaluate the kinetic energy of the flow, the internal energy of the compressed liquid and the work done by the external force in opposing the hydrodynamic force on the moving body. In particular we study the dependence of the energy distribution on the details of the body motion during the acceleration stage.

There are four relevant time scales in the problem: $T_a$, $T_c = R/c_0$, $T_d = R/V$, and $T_g = V/g$, where $c_0$ is the sound velocity in the liquid at rest, and $g$ is the acceleration due to gravity. Surface tension is considered to be negligible. $T_a$ is the duration of the acceleration stage. The second time scale, $T_c$, is associated with the effects of the liquid compressibility, and the third, $T_d$, with the body displacement. The fourth, $T_g$, corresponds to the effect of gravity acting on the free surface. In many practical problems the impact velocity $V$ is much less than $c_0$ so that the Mach number $M = V/c_0$ is small and, correspondingly, $T_c/T_d \ll 1$. We denote the ratio $T_c/T_a$ by $\alpha$ and assume that $\alpha = O(1)$, which means that we are dealing with violent motions of the body. Moreover, the flow is considered so violent that the ratio $T_a/T_g$ is also small and gravity may be neglected: that is the magnitude of the body acceleration is very much greater than $g$.

On the time scale $T_d$ both the displacement of the body and the free-surface elevation are visible: they are of the order of the body dimension. The flow has developed by this stage: close to the body it is incompressible at the leading order as $M \to 0$ and decays with the distance from the body. The flow is essentially nonlinear and usually can only be determined with the help of numerical methods. The hydrodynamic loads on the moving body are $O(\rho V^2)$, where $\rho$ is the liquid density, and are much less than the loads associated with the impact. This stage of the process is referred to as the stage of developed flow. Acoustic effects during this stage are localized near the disturbance front, which is generated at the moment of impact and propagates away from the body surface at the velocity of sound, $c_0$. Acoustic flow in the vicinity of the disturbance front gives a negligible contribution to both the velocity field and the pressure distribution in the main bulk of the liquid but its contribution to the energy of the liquid depends on details of the acceleration stage and can be significant. In any case we conclude that the asymptotic behaviour of the developed flow as $M \to 0$ is not uniform. Acoustic effects are of major significance near the disturbance front for all time for the ideal liquid without dissipative processes that is assumed.

The flow in the vicinity of the disturbance front depends on the characteristics of the initial stage of the impact, in particular on the body motion during the acceleration stage. Hence both the liquid flow and the pressure distribution, during the initial stage when $t'/T_d \ll 1$, are of importance for the further evolution of the process. Asymptotic solution of the problem as $t'/T_d \to 0$, $t'/T_a \to \infty$ and $M \to 0$ gives the initial data for the incompressible flow near the moving body and the flow near the acoustic disturbance front. The initial stage is of particular interest because that is when the hydrodynamic loads take their maximum values.

During the initial stage of motion the body displacement is small. As a result the equations of motion and the boundary conditions are linearized about the original rest state and the boundary conditions are taken at the initial position of the liquid boundary. The flow during the initial stage is irrotational and is described by a velocity potential, which satisfies the wave equation in the liquid domain, is equal to zero on the free surface and its normal derivative is equal to the normal velocity of the body on the wetted part of the rigid surface. Initial conditions for the wave equation are that the liquid is initially at rest.

For a body of an arbitrary shape, the solution of the problem can only be found numerically. The advantages of choosing the particular shape of a half-submerged sphere are (i) during the initial stage analytical forms for both the flow and the pressure distribution can be found for arbitrary body motion; (ii) both the kinetic and the internal energy of the liquid can be easily evaluated and analysed in detail; (iii) the boundary conditions on the body surface and liquid free surface correspond to each other near the contact line, this gives a solution which is regular at the contact line, with the advantage that there is less likelihood of jet formation at the initial stage. As noted above, this asymptotic problem is identical to the case of sudden motion of a sphere in an unbounded medium.

The major focus of this paper is the distribution of energy in the flow and its dependence on the details of the motion of the body during the acceleration stage. Although we have chosen an example where free-surface effects are minimized, this paper: (i) shows that acoustic effects can be of importance even for low impact velocities; (ii) helps to interpret the results given by the pressure-impulse theory; and (iii) demonstrates the way, in which the acoustic effects occur.

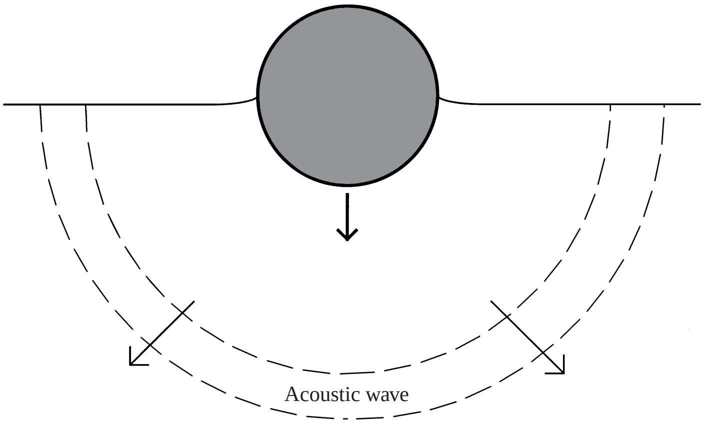


FIGURE 1. A sketch of the sphere, and the position of the acoustic zone at a late time.

The general description of the problem of impact on a half-submerged sphere and details of the pressure-impulse approach are given in § 2. The mathematical formulation of the problem is briefly presented in § 3. The velocity field and the pressure distribution are analysed in § 4 for an arbitrary law of the sphere motion. The method introduced by Kirchhoff (1876) and Love (1905) is used rather than the method of Longhorn (1952) which is more general but more complicated. The main result of § 4 is that for large times the region of almost incompressible flow close to the sphere and the region of the acoustic flow, which is attached to the compression wave front, are separated by an 'intermediate' region, where the liquid is effectively at rest. This result is very important because it explains the success of the pressure-impulse theory in impact problems. The temporal and spatial distribution of the fluid energy and its components are investigated in § 5. The analysis is similar to that performed by Brentner (1993) for nonlinear flows of a compressible fluid. However, within the linear acoustic approximation it is possible, even for an arbitrary motion of the sphere, to derive analytical formulae for the energy components of different parts of the flow region. This makes it possible to study the energy distribution in more detail. The global energy balance in the case of impulsive start of a sphere agrees with Longhorn (1952). The body motion that minimizes the radiated acoustic energy during the acceleration stage of a given duration to a given final velocity is obtained and compared with the case of uniform acceleration in § 6. Section 7 complements these prescribed velocities of the body with a brief discussion of Taylor's (1942) result which gives the free motion of a body after impact.

## 2. Impact on a half-submerged sphere

The problem of unsteady liquid flow caused by an impact on a sphere of radius $R$ is considered. Initially the sphere is half-submerged and the liquid is at rest. At the initial moment, $t' = 0$, the sphere starts to move down at a given velocity (figure 1). The liquid flow and the pressure distribution during the initial stage of the impact are determined under the following assumptions: (i) the sphere is rigid and undeformable; (ii) the liquid is ideal and compressible; (iii) gravity and surface tension are negligible; (iv) the sphere's velocity is given as $Vf(t'/T_a)$, where $0 \leqslant f(t'/T_a) \leqslant 1$ for $t' \geqslant 0$ with $f \equiv 1$ for $t' \geqslant T_a$, and where $T_a$ is the acceleration time; (v) the Mach number $M = V/c_0$ is much less than unity; (vi) the sphere displacement during the acceleration stage is very much less than the sphere radius $R$; (vii) the duration of the acceleration stage $T_a$ is comparable with the time scale for the acoustic stage, which is $O(R/c_0)$. The ratio $R/(c_0 T_a)$ is denoted by $\alpha$.

A shock wave is generated at the initial moment if $f(0) \neq 0$. During the initial stage when the shock is not far away from the sphere, the compressibility of the liquid is of major significance. If the body starts to move gradually, $f(0) = 0$, a shock wave is not formed but the liquid compressibility may still be important if the acceleration time interval is sufficiently short.

Provided that $M \ll 1$ a full description of the liquid flow during the initial stage can be found by using an acoustic approximation. This approximation is valid while deformation of the liquid domain, which can be estimated as $O(Vt')$, is small compared to the overall characteristic length $R$.

Within the framework of the acoustic approximation the boundary conditions can be taken on the undisturbed initial position of the liquid boundary and, moreover, both the equations of motion and the boundary conditions can be linearized about the initial rest state. The acoustic solution depends on the parameter $\alpha$, which indicates the magnitude of the body acceleration. Small values of $\alpha$ correspond to the case when the body velocity increases gradually and the acoustic effects are negligible at the end of the acceleration stage. Large values of $\alpha$ correspond to an almost instantaneous increase of the body velocity up to its maximum value which is unity in the dimensionless system. The limiting case, $\alpha \to \infty$, corresponds to an impulsive impact on the half-sphere floating on the liquid free surface. Both the acoustic effects, which are connected with the compressibility of the liquid, and the effects connected with the body acceleration are expected to decay as time increases. Under the assumptions listed above the contribution of both effects to the flow pattern becomes negligible almost everywhere well before the body displacement is significant. For this intermediate stage the liquid flow may be easily found with the pressure-impulse approach.

In the pressure-impulse approach we integrate the momentum equation in time from 0 to $t'_i$, where $T_a \ll t'_i \ll R/V$. Assuming $t'_i$ small and neglecting the integrals of the convective acceleration terms, we find (Lamb 1932) that

$$\boldsymbol{u}'(\boldsymbol{x}', t'_i) \approx -\frac{1}{\rho}\nabla P',$$

where $\boldsymbol{u}'(\boldsymbol{x}', t'_i)$ is the velocity vector of the liquid particles at the moment $t' = t'_i$, $P'(\boldsymbol{x}', t'_i)$ is the pressure impulse,

$$P'(\boldsymbol{x}', t'_i) = \int_0^{t'_i} p'(\boldsymbol{x}', \tau)\mathrm{d}\tau,$$

$p(\boldsymbol{x}', t')$ is the hydrodynamic pressure, $\rho$ is the liquid density, the scale of the pressure impulse is $\rho VR$. Assuming that the acoustic effects give a negligible contribution to the velocity field at $t' = t'_i$, we obtain

$$\nabla \cdot \boldsymbol{u}'(\boldsymbol{x}', t'_i) \approx 0.$$

After simple manipulations and neglecting the deformation of the flow domain during the initial stage, we arrive at the following boundary-value problem with respect to the pressure impulse:

$$\nabla^2 P' = 0 \qquad \text{(in the flow domain)},$$

$$P' = 0 \qquad \text{(on the free surface)},$$

$$\frac{\partial P'}{\partial n} = -\rho \boldsymbol{u}'_b \cdot \boldsymbol{n} \qquad \text{(on the sphere surface)},$$

where $\boldsymbol{u}'_b$ is the velocity of the body, and $\boldsymbol{n}$ is the unit normal vector to the body surface (Cooker & Peregrine 1995). We expect the velocity field given by the pressure-impulse theory to agree with the asymptotic value of the velocity derived from the acoustic theory below as $t \to \infty$ near the body.

## 3. Formulation of the problem

The dimensional variables used above, and identified by a prime, are now replaced with dimensionless variables using the sphere radius $R$ as length scale, the ratio $R/c_0$ as the time scale but scaling the liquid velocity with $V$, and using the 'water hammer' pressure $\rho c_0 V$ to scale pressure.

The spherical coordinates $r, \varphi, \theta$ are introduced with the origin at the sphere centre, $\varphi$ the longitude, $-\pi < \varphi \leqslant \pi$, and $\theta$ the polar angle, $0 \leqslant \theta \leqslant \pi$, measured from the lowest point of the sphere. The spherical and Cartesian coordinates are related by $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$. The flow domain coincides with that occupied by the liquid at the initial moment, $r \geqslant 1$, $0 \leqslant \theta \leqslant \pi/2$, $-\pi < \varphi \leqslant \pi$. The liquid flow is axisymmetric and described by the velocity potential $\Phi(r, \theta, t)$, for which the initial boundary-value problem has the form

$$\Phi_{tt} = \nabla^2 \Phi \qquad (r > 1,\ 0 \leqslant \theta < \pi/2), \tag{3.1}$$

$$\Phi = 0 \qquad (r > 1,\ \theta = \pi/2), \tag{3.2}$$

$$\Phi_r = f(\alpha t)\cos\theta \qquad (r = 1,\ 0 \leqslant \theta < \pi/2), \tag{3.3}$$

$$\Phi = \Phi_t = 0 \qquad (t < 0). \tag{3.4}$$

Once problem (3.1)–(3.4) has been solved, the velocity field $\boldsymbol{u} = (u_r, u_\theta)$, $u_r = \Phi_r$, $u_\theta = r^{-1}\Phi_\theta$, and the pressure $p = -\Phi_t$ can be evaluated. The solution $\Phi(r, \theta, t)$ is sought among the functions defined in the region $\Omega = \{r, \theta \mid r \geqslant 1,\ 0 \leqslant \theta \leqslant \pi/2\}$ and describing flows with both kinetic and potential energies finite. The kinetic $T$ and potential $\Pi$ energies of the flow are defined in the spherical coordinates by

$$T = 3\int_1^\infty r^2\,\mathrm{d}r \int_0^{\pi/2} (\boldsymbol{u})^2 \sin\theta\,\mathrm{d}\theta, \tag{3.5}$$

$$\Pi = 3\int_1^\infty r^2\,\mathrm{d}r \int_0^{\pi/2} p^2 \sin\theta\,\mathrm{d}\theta, \tag{3.6}$$

with the product $m_a V^2$ being the energy scale, where $m_a = \frac{1}{3}\pi\rho R^3$ is the added mass of the half-submerged sphere.

The energy conservation law in the acoustic approximation can be derived by multiplication of the wave equation (3.1) by $\Phi_t$ and integration of the result over the liquid domain. After some manipulation we obtain

$$\begin{aligned} \frac{\pi}{3}\frac{\mathrm{d}}{\mathrm{d}t}[T(t) + \Pi(t)] &= \iint_S p(1, \theta, t)\Phi_r(1, \theta, t)\mathrm{d}S \\ &= f(\alpha t)\iint_S p(1, \theta, t)\cos\theta\,\mathrm{d}S = f(\alpha t)F(t), \end{aligned} \tag{3.7}$$

where $S$ is the hemisphere's surface and $F(t)$ is the total hydrodynamic force on the moving sphere. After integration with respect to time and using the initial conditions (3.4), we obtain

$$T(t) + \Pi(t) = A(t), \tag{3.8}$$

where

$$A(t) = \frac{3}{\pi}\int_0^t f(\alpha\tau)F(\tau)\,\mathrm{d}\tau \tag{3.9}$$

is the dimensionless work done to overcome the liquid's resistance to the sphere motion. In dimensional variables,

$$A' = m_a V^2 A(t) = \int_0^{t'} V f(\tau'/T_a)F'(\tau')\,\mathrm{d}\tau',$$

where $F'(t') = \rho c_0 V R^2 F(t'c_0/R)$ is the dimensional hydrodynamic force. The dimensionless energy conservation equation (3.8) is used below to analyse the energy distribution over the liquid region and the energy evolution in time.

The disturbance front, $r = t + 1$, is spherical in this problem; the liquid is disturbed when $1 < r < t + 1$ and is at rest when $r > t + 1$. The initial conditions (3.4) give $\Phi(r, \theta, t) \equiv 0$ when $r > t + 1$ and, therefore, the integration in equations (3.5)–(3.6) is only from $r = 1$ to $r = t + 1$.

The pressure impulse for the incompressible flow initiated by the sphere's motion is easily found to be

$$P(r, \theta) = \frac{1}{2r^2}\cos\theta, \tag{3.10}$$

the scale of the pressure impulse being $\rho V R$, with the components of the velocity vector being

$$\hat{u}_r = -P_r(r, \theta), \qquad \hat{u}_\theta = -r^{-1}P_\theta(r, \theta). \tag{3.11}$$

The quantities derived from pressure-impulse theory are denoted by a hat. Therefore, we can evaluate the kinetic energy of the flow predicted by the pressure-impulse theory but it gives no information about the pressure magnitude and the work done to overcome the hydrodynamic force on the moving body.

## 4. Velocity field and the pressure distribution

The acoustic velocity potential of the flow can be taken to have the form $\Phi(r, \theta, t) = \phi(r, t)\cos\theta$. Substitution of this representation into the wave equation (3.1) written in spherical coordinates

$$\Phi_{tt} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)$$

gives the equation for $\phi(r, t)$

$$r^2\phi_{tt} = (r^2\phi_r)_r - 2\phi \qquad (r > 1, t > 0). \tag{4.1}$$

The boundary and initial conditions for equation (4.1) follow from (3.3), (3.4) and are

$$\phi_r = f(\alpha t) \qquad (r = 1, t > 0), \tag{4.2}$$

$$\phi = \phi_t = 0 \qquad (t < 0). \tag{4.3}$$

The solution of the boundary-value problem (4.1)–(4.3) is sought in the functional space $\phi(r, \cdot) \in H^1(1, \infty)$, $\phi_t(r, \cdot) \in L_2(1, \infty)$, which yields finite values for the kinetic and potential energies of the flow. This means, in particular, that $\phi(r,\cdot)$ is continuous where $r \geqslant 1$ at any fixed time moment $t$.

Equation (4.1) is satisfied by the function (Kirchhoff 1876)

$$\phi(r,t) = \frac{\partial}{\partial r}\left(\frac{1}{r}\psi(\xi)\right), \tag{4.4}$$

where $\xi = t - r + 1$ and $\psi(\xi)$ is an arbitrary function. We obtain

$$\phi(r,t) = -\frac{1}{r^2}\psi - \frac{1}{r}\psi_\xi. \tag{4.5}$$

Equations (4.3) and the restriction that $\phi(r,t)$ is a continuous function of $r$ at any fixed instant $t$ yield that $\psi(\xi)$ and $\psi_\xi(\xi)$ are continuous for all $\xi$.

The boundary condition (4.2) on the sphere surface and the initial conditions (4.3) yield the following equations for $\psi(\xi)$ (see Kirchhoff 1876):

$$\left.\begin{aligned} \psi_{\xi\xi} + 2\psi_\xi + 2\psi = f(\alpha\xi) \qquad & (\xi > 0), \\ \psi(\xi) \equiv 0 \qquad & (\xi < 0). \end{aligned}\right\} \tag{4.6}$$

The value $\xi = 0$ corresponds to the front of the disturbances, $r = t + 1$. Ahead of the moving disturbance front, $r > t + 1$, the liquid is at rest. Continuity of the functions $\psi(\xi)$ and $\psi_\xi(\xi)$, where $-\infty < \xi < +\infty$, gives

$$\psi(0+) = \psi_\xi(0+) = 0. \tag{4.7}$$

The last two equations are initial conditions for equation (4.6), which makes it possible to determine the unknown function $\psi(\xi)$ by quadratures:

$$\psi(\xi) = \int_0^\xi f(\alpha\xi_0)\mathrm{e}^{\xi_0-\xi}\sin(\xi - \xi_0)\,\mathrm{d}\xi_0. \tag{4.8}$$

Once the function $\psi(\xi)$ has been evaluated, the radial $u_r$ and the angular $u_\theta$ components of the liquid velocity are given by

$$u_r(r,\theta,t) = (r^{-1}\psi_{\xi\xi} + 2r^{-2}\psi_\xi + 2r^{-3}\psi)\cos\theta, \tag{4.9}$$

$$u_\theta(r,\theta,t) = (r^{-2}\psi_\xi + r^{-3}\psi)\sin\theta, \tag{4.10}$$

and the pressure distribution $p(r,\theta,t)$ by

$$p(r,\theta,t) = (r^{-1}\psi_{\xi\xi} + r^{-2}\psi_\xi)\cos\theta. \tag{4.11}$$

In particular, on the rigid surface

$$\left.\begin{aligned} & p(1,\theta,t) = p(1,0,t)\cos\theta, \\ & p(1,0,t) = -\phi_t(1,t) = \psi_{\xi\xi}(t) + \psi_\xi(t). \end{aligned}\right\} \tag{4.12}$$

Taking (4.6) into account, we obtain from (4.12)

$$p(1,\theta,t) = f(\alpha t)\cos\theta - [\psi_\xi(t) + 2\psi(t)]\cos\theta, \tag{4.13}$$

where the term $f(\alpha t)\cos\theta$ corresponds to the geometrical acoustic approximation. The second term on the right of (4.13) can be neglected compared with the first one for small times only, $t \ll 1$. There is no particular function $f(\alpha t)$ for which this term is negligible for all times. This may be seen from the equation $\psi_\xi(t) + 2\psi(t) = 0$ which gives $\psi(t) = C\exp(-2t)$, where $C = 0$ owing to the initial conditions (4.7); then equation (4.6) predicts $f(\alpha\tau) \equiv 0$ for $\psi(\tau) = 0$, which corresponds to the rest state.

The pressure $p(1, 0, t)$, at the base of the sphere is not uniformly positive but changes sign. In order to demonstrate this for an arbitrary law of the sphere motion, consider $t > 1/\alpha$, where $1/\alpha$ is the duration of the acceleration stage in the dimensionless variables. For $t > 1/\alpha$ the body velocity $f(\alpha t)$ is equal to unity and equation (4.6) gives

$$\psi(t) = C_1 \mathrm{e}^{-t} \sin t + C_2 \mathrm{e}^{-t} \cos t + \tfrac{1}{2}, \tag{4.14}$$

where the constants $C_1$ and $C_2$ are determined by matching this solution with the solution of equation (4.6) with (4.7) for $0 < t < 1/\alpha$. In the general case, these constants are arbitrary. From (4.12)

$$\left.\begin{aligned} p(1, 0, t) &= \mathrm{e}^{-t}[(C_2 - C_1) \sin t - (C_2 + C_1) \cos t] \\ p(1, 0, 2\pi N + \pi) &= -\mathrm{e}^{-\pi} p(1, 0, 2\pi N), \end{aligned}\right\} \tag{4.15}$$

where $N$ is integer, $N > (2\pi\alpha)^{-1}$. Therefore, the pressure oscillates as $t > 1/\alpha$ with its magnitude decaying exponentially with time.

For the case of the impulsive start of the sphere, $T_a = 0$, we get $f(\alpha t) \equiv 1$ for $t > 0$, and the constants are $C_1 = C_2 = -1/2$. The pressure evolution at the base of the sphere follows from (4.15):

$$p(1, 0, t) = \mathrm{e}^{-t} \cos t. \tag{4.16}$$

The negative pressure on the sphere surface peaks at $t = \frac{3}{4}\pi$ with its magnitude being $2^{-1/2} \exp(-3\pi/2)$. This means that the magnitude of the negative pressures, which occur on the sphere, is less than $6.35 \times 10^{-3}$ of the 'water hammer' pressure $\rho V c_0$. In particular, for the impulsive impact on a sphere floating on the water surface, $\rho = 1000\,\mathrm{kg\,m^{-3}}$, $c_0 = 1500\,\mathrm{m\,s^{-1}}$, given a velocity $V = 2\,\mathrm{m\,s^{-1}}$ the pressure on the sphere peaks at the impact instant, $t = 0$, with its value being 3 MPa, and cannot be lower than −19 KPa.

The hydrodynamic force on the sphere $F(t)$ is given by

$$F(t) = \iint_S p(1, \theta, t) \cos\theta \,\mathrm{d}S = -\phi_t(1, t) \iint_S \cos^2\theta \,\mathrm{d}S,$$

where $S$ is the wetted part of the sphere surface, the scale of the force being $\rho c_0 V R^2$. The last integral equals half the volume of a hemisphere of unit radius, which is $\frac{2}{3}\pi$. Hence

$$F(t) = -\tfrac{2}{3}\pi\phi_t(1, t) = \tfrac{2}{3}\pi p(1, 0, t), \tag{4.17}$$

where $p(1, 0, t)$ is given by equation (4.12). Therefore, the hydrodynamic force $F(t)$ is proportional to the pressure at the sphere base and they oscillate together.

The liquid particles of the free surface, $\theta = \pi/2$, only move vertically, which follows from (4.9) and (4.10). Their vertical velocity is given by

$$u_\theta(r, \tfrac{1}{2}\pi, t) = \frac{1}{r^2}\psi_\xi(t - r + 1) + \frac{1}{r^3}\psi(t - r + 1) \qquad (r > 1).$$

The elevation of the free surface is described by the equation $z = \zeta(r, t)$, $r > 1$, where the function $\zeta(r, t)$ can be obtained by integration of the vertical velocity in time. We find

$$\zeta(r, t) = \frac{1}{r^2}\psi(t - r + 1) + \frac{1}{r^3}\int_0^{t-r+1} \psi(\xi)\mathrm{d}\xi \qquad (1 < r < t + 1),$$

$$\zeta(r, t) \equiv 0 \qquad (r > t + 1),$$

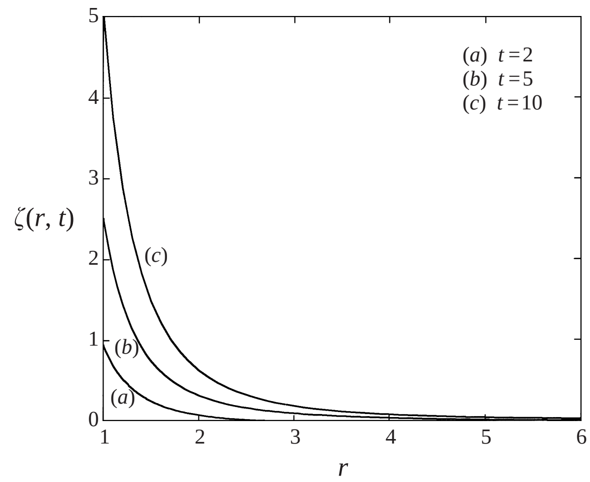


FIGURE 2. The dimensionless free-surface displacement for the case of an impulsive start.

or, using equation (4.6),

$$\zeta(r,t) = \left(\frac{1}{r^2} - \frac{1}{r^3}\right)\psi(t-r+1) - \frac{1}{2r^3}\psi_\xi(t-r+1) + \frac{1}{2r^3}\int_0^{t-r+1} f(\alpha\tau)\mathrm{d}\tau.$$

It is seen that

$$\zeta(1,t) = \tfrac{1}{2}\int_0^t f(\alpha\tau)\mathrm{d}\tau + O(\mathrm{e}^{-t}) \qquad (t\to\infty).$$

That is, the maximum elevation of the free surface tends exponentially with time to half the depth of sphere penetration. For the impulsive start of the sphere we find

$$\zeta(r,t) = \frac{t}{2r^3} + \tfrac{1}{2}\mathrm{e}^{-(t-r+1)}\left[\frac{1}{r^3}\cos(t-r+1) - \frac{1}{r^2}(\sin(t-r+1)+\cos(t-r+1))\right]. \tag{4.18}$$

The shape of the free surface is shown in figure 2. The first term in (4.18) corresponds to the free-surface elevation predicted by the pressure-impulse theory (see (3.11)). The second term decays exponentially with increasing time except in a narrow zone near the disturbance front, $r = t + 1$.

Asymptotics of the velocity components $u_r(r,\theta,t)$ and $u_\theta(r,\theta,t)$ as $t\to\infty$ are non-uniform. Equations (4.9) and (4.10) predict that

$$u_r(r,\theta,t) = \frac{\cos\theta}{r^3} + O(\mathrm{e}^{-t}),\;\; u_\theta(r,\theta,t) = \frac{\sin\theta}{2r^3} + O(\mathrm{e}^{-t}), \tag{4.19}$$

close to the body, $r = O(1)$, $\xi = O(t)$, and

$$u_r(r,\theta,t) = \frac{1}{r}\psi_{\xi\xi} + O(t^{-2}),\;\; u_\theta(r,\theta,t) = \frac{1}{r^2}\psi_\xi + O(t^{-3}), \tag{4.20}$$

close to the disturbance front, $r = O(t)$, $\xi = O(1)$, as $t\to\infty$. The leading terms in (4.19) correspond to the velocity field predicted by the pressure-impulse theory (see (3.11)).

In order to illustrate the non-uniform distribution of the velocity and the pressure as $t\to\infty$, the products $r\,u_r(r,0,t)$ and $r\,p(r,0,t)$ in the case of impulsive start of the sphere, where $r = t\beta + 1$, $0 < \beta < 1$, are shown in figure 3 as functions of $\beta$ for

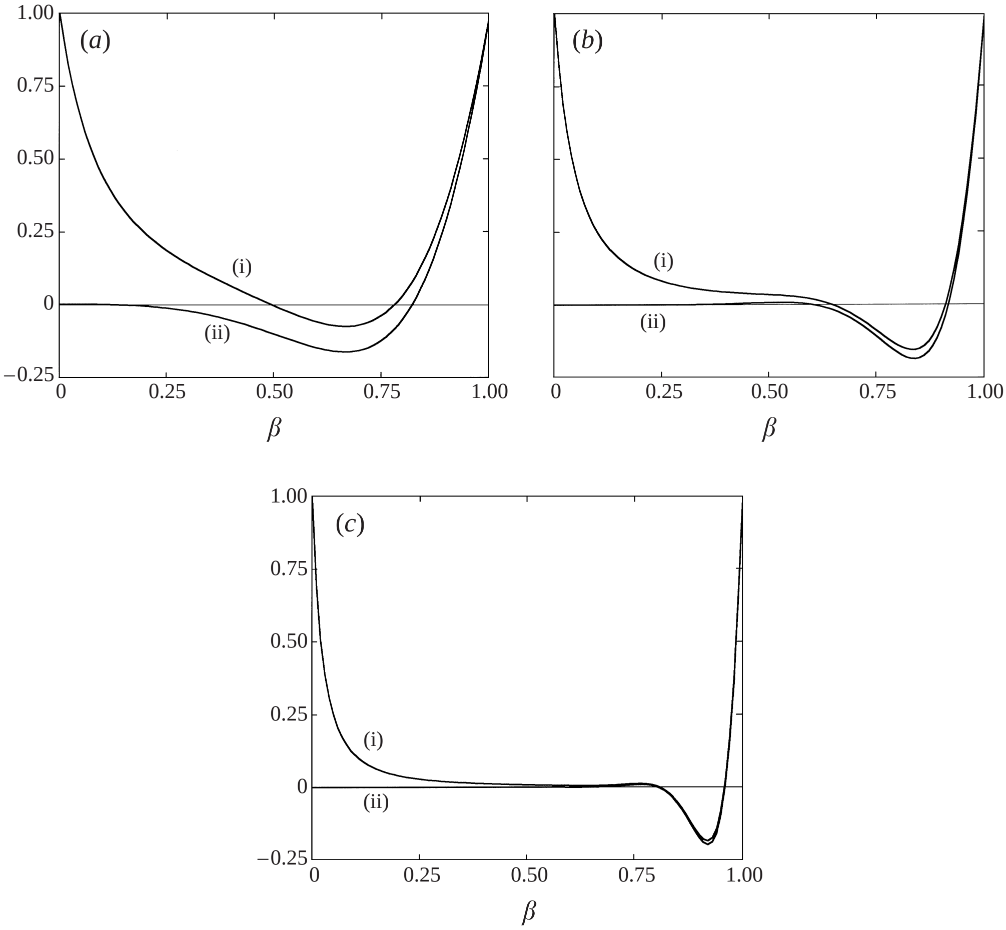


FIGURE 3. For impulsive motion of a sphere: the radial component of velocity, line (i), and the pressure, line (ii), each multiplied by $r$, at ($a$) $t = 5$, ($b$) $t = 10$, ($c$) $t = 20$.

different time instants. It is seen that the two regions, where $r = O(1)$ and where $\xi = O(1)$, are separated by an 'intermediate' region, where the liquid is almost at rest. In the acoustic region, $\xi = O(1)$, we obtain $p(r, 0, t) \approx u_r(r, 0, t)$, which indicates that the flow in the vicinity of the acoustic front is a hemispherical outward-propagating acoustic wave.

Formula (4.11) for the pressure distribution shows that the pressure $p(r, \theta, t)$ tends to zero with increasing time at every fixed point $r$. At a fixed instant the pressure peaks close to the disturbance front and decays exponentially with the distance away from it. The pressure is related entirely to the acoustic wave: the pressure of the incompressible flow is $O(\rho V^2)$ and negligible in this linearized analysis.

## 5. The energy conservation law

The work $A(t)$ done to oppose the pressure force is given by (3.9), which can be written, using (4.17), in the form

$$A(t) = -2 \int_0^t f(\alpha\tau)\phi_t(1, \tau)\,\mathrm{d}\tau. \tag{5.1}$$

The kinetic energy of the flow $T$ is given by

$$T(t) = \int_1^{t+1} [\phi_r^2(r,t)r^2 + 2\phi^2(r,t)]\mathrm{d}r \tag{5.2}$$

and the potential energy of compressed liquid $\Pi(t)$ by

$$\Pi(t) = \int_1^{t+1} \phi_t^2(r,t)r^2\mathrm{d}r. \tag{5.3}$$

The integration with respect to $r$ is carried out from the sphere surface, $r = 1$, to the disturbance front, $r = t + 1$, because $\phi(r,t) \equiv 0$ where $r > t + 1$. The energy conservation law (3.8) can now be rewritten as

$$\int_1^{t+1} (r^2\phi_t^2 + r^2\phi_r^2 + 2\phi^2)\,\mathrm{d}r = -2\int_0^t f(\alpha\tau)\phi_t(1,\tau)\,\mathrm{d}\tau \tag{5.4}$$

which can also be obtained from the boundary-value problem (4.1)–(4.3) for the function $\phi(r,t)$.

Physical reasoning (see §4 and Brentner 1990, 1993) shows that both the kinetic $T(t)$ and the potential $\Pi(t)$ energies of the liquid are distributed non-uniformly over the flow region. In order to analyse the evolution of the energy components in time and their distributions in space, we introduce new functions

$$E_K(r,t) = \int_r^{t+1} [\phi_r^2(r,t)r^2 + 2\phi^2(r,t)]\mathrm{d}r, \tag{5.5}$$

$$E_\Pi(r,t) = \int_r^{t+1} \phi_t^2(r,t)r^2\mathrm{d}r, \tag{5.6}$$

where $E_K(1,t) = T(t)$, $E_\Pi(1,t) = \Pi(t)$. The functions $E_K(r_*,t)$ and $E_\Pi(r_*,t)$ describe the kinetic energy and the potential energy of the liquid in the region $r_* < r < t+1$, $0 < \theta < \pi/2$, which is attached to the disturbance front, at an instant $t$.

Some manipulation yields

$$E_\Pi(r,t) = \tfrac{1}{2}A(\xi) - S(\xi) + \left(\frac{1}{r} - 1\right)\psi_\xi^2(\xi), \tag{5.7}$$

$$E_K(r,t) = \tfrac{1}{2}A(\xi) - \left(1 - \frac{2}{r^2}\right)S(\xi) - \left(1 - \frac{1}{r}\right)^2 \psi_\xi^2(\xi). \tag{5.8}$$

where

$$S(t) = \tfrac{1}{2}\psi_\xi^2(t) + \psi^2(t) + 2\psi(t)\psi_\xi(t), \tag{5.9}$$

giving

$$T(t) = E_K(1,t) = \tfrac{1}{2}A(t) + S(t), \tag{5.10}$$

$$\Pi(t) = E_\Pi(1,t) = \tfrac{1}{2}A(t) - S(t). \tag{5.11}$$

Equations (5.10) and (5.11) clearly show that the energy conservation law (3.8) is satisfied.

For pressure-impulse theory the compression of the liquid under the impact is not taken into account, therefore $\hat{\Pi} = 0$. The kinetic energy $\hat{T}$ predicted by this theory is equal to $\frac{1}{2}$, from equations (3.11) and (3.5). The scale of the energy is the same, $m_aV^2$, as in the acoustic approach (see §3). The quantity, which in the pressure-impulse theory plays a role similar to the work done, $A(t)$, to oppose the hydrodynamic force on the sphere, is the total impulse $P'_{tot}$: it is the integral of the pressure impulse $P'(r',\theta)$ multiplied by the normal component of the velocity of the sphere surface, $V\cos\theta$, over the wetted part of the sphere

$$P'_{tot} = \iint_{S'} P'(r',\theta) V \cos\theta \, \mathrm{d}s' \\ = \rho V^2 R^3 \iint_S P(1,\theta)\cos\theta \, \mathrm{d}s.$$

In the dimensionless variables, $P_{tot} = 1$, which follows from (3.10), with the scale of the total impulse being $m_a V^2$.

The function $E_\Pi(r,t)$ depends on the variable $\xi$, which is the distance from the disturbance front. Equations (4.6), (5.1) and (5.9) give

$$\psi_\xi^2(\xi) = O(\mathrm{e}^{-2\xi}), \quad S(\xi) = \tfrac{1}{4} + O(\mathrm{e}^{-\xi}), \quad A(\xi) = A(\infty) + O(\mathrm{e}^{-\xi}) \tag{5.12}$$

as $\xi \to \infty$, $t \to \infty$. Taking (5.12) into account, we obtain that

$$E_\Pi(r,t) = \tfrac{1}{2}A(\infty) - \tfrac{1}{4} + O(\mathrm{e}^{-\xi}) \tag{5.13}$$

as $t \to \infty$ and $t - r \to \infty$ . Equation (5.13) confirms that the main contribution to the potential energy comes from the vicinity of the disturbance front.

In the same manner we can find from (5.8) that the kinetic energy of the flow $T(t) - E_K(r,t)$ near the body, $r = O(1)$, and for large time, $t \gg 1$, is given by

$$T(t) - E_K(r,t) = \frac{1}{2} - \frac{1}{2r^2} + O(\mathrm{e}^{-t}). \tag{5.14}$$

In particular,

$$T(t) - E_K(r,t) = \tfrac{1}{2} + o(1)$$

as $t \to \infty$, $r \to \infty$ and $r/t \to 0$. The last limit corresponds to the kinetic energy $\hat{T}$ given by the pressure-impulse theory. The total kinetic energy $T(t)$ tends to $\frac{1}{2}A(\infty) + \frac{1}{4}$ as $t \to \infty$, which follows from (5.10) and (5.12). Therefore, the kinetic energy concentrated near the disturbance front, $E_K(r,t)$, tends to $\frac{1}{2}A(\infty) - \frac{1}{4}$ as $t \to \infty$. It is seen that both the kinetic and potential energies carried by the acoustic wave are equal. Therefore the total energy $E_{ac}$ taken away with the acoustic wave is $A(\infty) - \frac{1}{2}$ and the kinetic energy of the flow near the moving body is $\frac{1}{2}$ for $t \gg 1$. The total energy of the flow is equal to $A(\infty)$ which is the total work done to oppose the hydrodynamic force. Thus in this example, with no significant free-surface motion, acoustic effects are responsible for the 'lost' energy on impact.

The effects of differing acceleration regimes are illustrated with the values of $E_\Pi(r,t)$ and $E_K(r,t)$ as functions of position at chosen time instants in figures 4 to 6 for the case $\alpha = 0.5$. Figure 4 corresponds to a fully impulsive start of the body motion, $f(\tau) = 1$, and figure 5 to a constant acceleration ($f(\tau) = \tau$, $0 < \tau < 1$). Figure 6 corresponds to a mixture of impulsive and constant acceleration: $f(\tau) = (\tau + 1)/2$. In all cases $f(\tau) = 1$ for $\tau > 1$. It is clearly seen that the potential energy of the compressed liquid is localized close to the disturbance front and its value is almost constant for $t > 4$. The kinetic energy is stabilized later, after $t = 6$. These figures demonstrate that the energy taken away with the acoustic wave and the work required to accelerate the body up to the given velocity depend on the details of the body acceleration. Comparing these figures, we may conclude that a constant acceleration is a more economical way to provide the body with a given velocity, but uniform acceleration is not the best way, as is shown in §6.

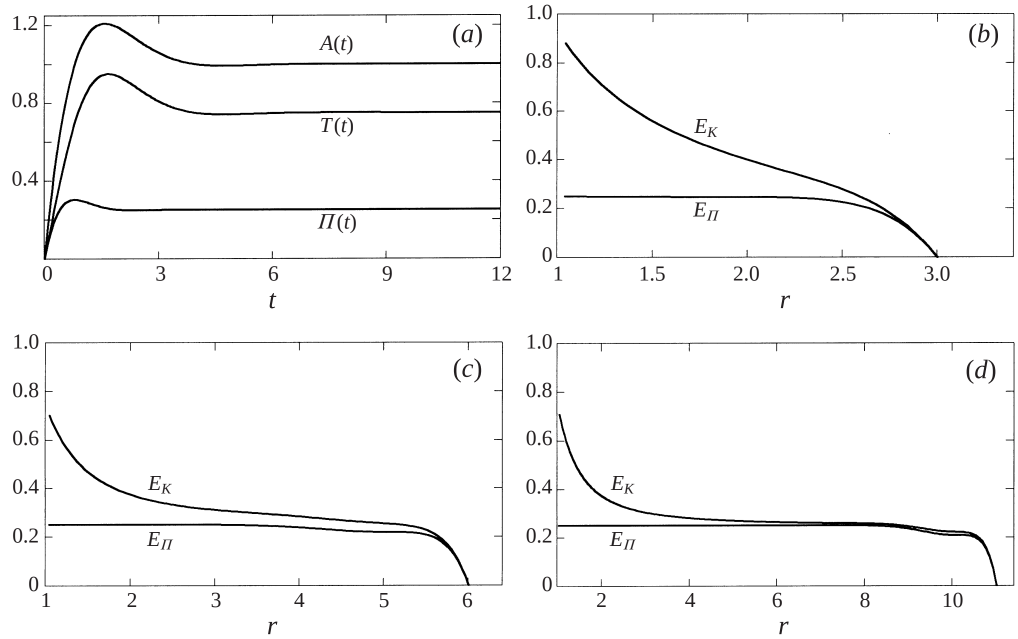


FIGURE 4. Impulsive motion of the sphere: ($a$) work done $A(t)$, kinetic energy $T(t)$, and potential energy of compression $\Pi(t)$. ($b$–$d$) Distributions of the kinetic energy, $E_K$, and the potential energy, $E_\Pi$, over the flow at ($b$) $t = 2$, ($c$) $t = 5$, ($d$) $t = 10$.

Equation (5.1) for the work $A(t)$ done to oppose the hydrodynamic force indicates that $A(t)$ takes its local extremum at the instants $t_j$ when the pressure $p(1, 0, t)$ at the sphere base is zero. The work $A(t)$ takes its local maximum value when the pressure $p(1, 0, t)$ changes from positive to negative and its local minimum value when the pressure changes from negative to positive at $t = t_j$. This means that the work $A(t)$ is not a monotonic function of time. In particular, for $t > 1/\alpha$ it can be presented as

$$A(t) = A_s + A_b(t),$$

where $A_s$ is the work done at the acceleration stage and $A_b(t)$ is the work done after the end of the acceleration stage until the instant $t$. Equation (5.1) gives

$$A_b(t) = -2\int_{1/\alpha}^{t} \phi_\tau(1, \tau)\mathrm{d}\tau = 2[\phi(1, 1/\alpha) - \phi(1, t)]$$

and

$$A_s = -2\phi(1, 1/\alpha) + 2\alpha \int_0^{1/\alpha} \dot{f}(\alpha\tau)\phi(1, \tau)\mathrm{d}\tau,$$

where $\dot{f}(\alpha\tau)$ is the body acceleration. For the impulsive start of the sphere, $\alpha \to \infty$, we obtain $A_s \to 0$, $\phi(1, 1/\alpha) \to 0$ and

$$A(t) = -2\phi(1, t).$$

Taking into account equations (4.5) and (4.14), we find (see Longhorn 1952)

$$A(t) = 1 + \mathrm{e}^{-t}(\sin t - \cos t).$$

The work $A(t)$ takes its maximum value $1 + \exp(-\pi/2)$ at the moment $t = \pi/2$,

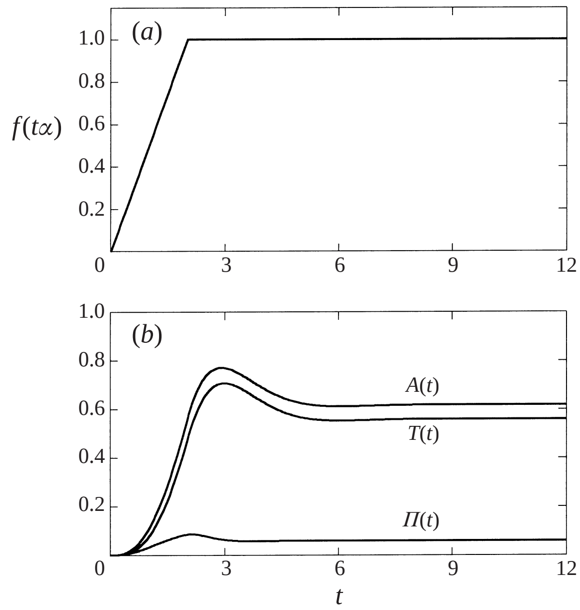


FIGURE 5. Uniform acceleration of the sphere: (*a*) dimensionless velocity of the sphere, (*b*) work done $A(t)$, kinetic energy $T(t)$, and potential energy of compression $\Pi(t)$.

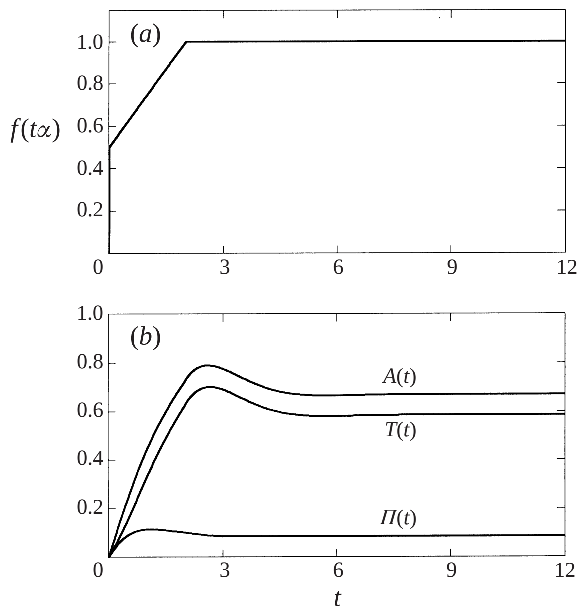


FIGURE 6. Partly impulsive and partly uniform acceleration of sphere: (*a*) dimensionless velocity of the sphere, (*b*) work done $A(t)$, kinetic energy $T(t)$, and potential energy of compression $\Pi(t)$.

when $p(1, 0, \pi/2) = 0$ (see figure 7), and oscillates with time thereafter. The function $A(t)$ decays in the time interval $(\pi/2, 3\pi/2)$ because the hydrodynamic loads are negative there and the body continues to move down. During the initial time interval $(0, \pi/2)$ the liquid has been accelerated so much that it pulls the sphere down for

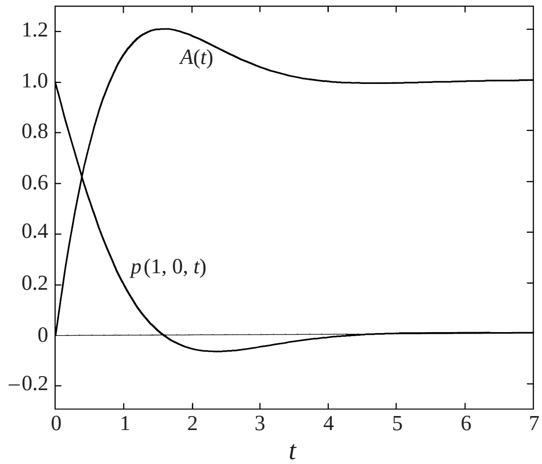


FIGURE 7. Work done, $A(t)$, and pressure at the base of the sphere for the case of an impulsive start.

$\pi/2 < t < 3\pi/2$ owing to the adhesive forces. It should be noted that the total work $A(\infty)$ in this case is equal to unity and is about 20% less than $A(\pi/2)$.

In general the total work is given by

$$A(\infty) = 1 + 2\alpha \int_0^{1/\alpha} \dot{f}(\alpha\tau)\phi(1,\tau)\mathrm{d}\tau, \tag{5.15}$$

where the first term corresponds to the total impulse $P_{tot} = 1$ and the integral term depends on a particular acceleration of the body. In the case of a gradual acceleration of the body, $\alpha \ll 1$, with $f(0) = 0$, $f'(0) = 0$, $f(1) = 1$, acoustic effects can be neglected, the energy $E_{ac}$ taken away with the disturbance front is much less than the kinetic energy of the incompressible flow $\hat{T}$ and the energy conservation law (3.8) predicts $A(\infty) = \frac{1}{2}$. Equation (5.15) gives the same value as $\alpha \to 0$. Taking (4.5) into account and introducing new variables $\tau = \sigma/\alpha$, $s(\sigma,\alpha) = \psi(\sigma/\alpha)$, we can rewrite equation (5.15) as

$$A(\infty) = 1 - 2\int_0^1 \dot{f}(\sigma)[s(\sigma,\alpha) + \alpha s_\sigma(\sigma,\alpha)]\,\mathrm{d}\sigma,$$

where $s(\sigma,\alpha)$ satisfies the equation

$$\alpha^2 s_{\sigma\sigma} + 2\alpha s_\sigma + 2s = f(\sigma) \qquad (0 < \sigma < 1),$$

which follows from (4.6). At leading order as $\alpha \to 0$, we find

$$A(\infty) = 1 - 2\int_0^1 \dot{f}(\sigma)s(\sigma,0)\mathrm{d}\sigma + O(\alpha), \quad s(\sigma,0) = \tfrac{1}{2}f(\sigma),$$

which yields

$$A(\infty) \approx 1 - \int_0^1 f(\sigma)\dot{f}(\sigma)\mathrm{d}\sigma = 1 - \tfrac{1}{2}[f^2(1) - f^2(0)] = \tfrac{1}{2}.$$

Therefore for smooth functions $f(\sigma)$, $-\infty < \sigma < \infty$ and small $\alpha$, the acoustic effects can be neglected with the accuracy being $O(\alpha)$. However, even for a relatively long acceleration stage ($\alpha \ll 1$) this incompressible liquid model can give wrong results if the body velocity $f(\alpha t)$ is not continuous in time.

For a given $\alpha$ the total work $A(\infty)$ is dependent on the acceleration history of the body for $0 \leqslant t \leqslant 1/\alpha$, the quantity $E_{ac} = A(\infty) - \frac{1}{2}$ being equal to the energy taken away with the acoustic wave. In the general case $(0 < \alpha < +\infty)$ we expect that the total work is limited from below by the kinetic energy $\hat{T}$ of the incompressible flow. The lower limit corresponds to a small acceleration parameter $\alpha$. A related property is considered in the next section: for given acceleration time $T_a$ how can the acoustic part $E_{ac}$ of the energy be minimized by an optimal choice of the body acceleration.

## 6. Optimal acceleration of floating body

The optimal acceleration that we seek gives the motion of the floating body, during the acceleration stage of a given duration, that minimizes the acoustic energy that is radiated. The acoustic part $E_{ac}$ of the total energy of the liquid is defined as $E_{ac} = A(\infty) - \hat{T}$, where $A(\infty)$ is the total work done to oppose the hydrodynamic force, which is given by (5.1), and $\hat{T}$ is the kinetic energy of the incompressible flow, which is given by the pressure-impulse theory and is equal to $\frac{1}{2}$. With the help of equations (5.1), (4.5) and (4.6) we obtain

$$E_{ac} = \frac{1}{2} - 2\left[\psi\left(\frac{1}{\alpha}\right) + \psi_\xi\left(\frac{1}{\alpha}\right)\right] + 2\int_0^{1/\alpha} f(\alpha\tau)[f(\alpha\tau) - 2\psi(\tau) - \psi_\xi(\tau)]\,\mathrm{d}\tau, \qquad (6.1)$$

and determine the body velocity $f(\alpha\tau)$ which minimizes the value $E_{ac}$.

It is convenient to introduce the following notation: $u(\tau) = f(\alpha\tau)$, $x_1(t) = \psi(t)$, $x_2(t) = \psi_\xi(t)$, $\boldsymbol{x}(t) = (x_1, x_2)$, $t_0 = 0$, $t_1 = 1/\alpha$, with which the problem can be rewritten in terms of optimal control:

$$\mathfrak{R}_0(\boldsymbol{x}, u) \to \min, \qquad (6.2)$$

with the additional restrictions

$$\dot{x}_1 = w_1(t, \boldsymbol{x}, u), \qquad \dot{x}_2 = w_2(t, \boldsymbol{x}, u), \qquad (6.3)$$

$$\mathfrak{R}_1(\boldsymbol{x}, u) = 0, \qquad \mathfrak{R}_2(\boldsymbol{x}, u) = 0, \qquad (6.4)$$

where

$$\mathfrak{R}_i(\boldsymbol{x}, u) = \int_{t_0}^{t_1} h_i(\tau, \boldsymbol{x}, u)\,\mathrm{d}\tau + \omega_i(t_0, \boldsymbol{x}(t_0), t_1, \boldsymbol{x}(t_1)) \quad (i = 0, 1, 2), \qquad (6.5)$$

$$h_0(t, \boldsymbol{x}, u) = 2u(u - 2x_1 - x_2), \qquad (6.6)$$

$$\omega_0(t_0, \boldsymbol{x}(t_0), t_1, \boldsymbol{x}(t_1)) = \tfrac{1}{2} - 2[x_1(t_1) + x_2(t_2)], \qquad (6.7)$$

$$h_1(t, \boldsymbol{x}, u) = 0, \qquad \omega_1 = x_1(t_0), \qquad (6.8)$$

$$h_2(t, \boldsymbol{x}, u) = 0, \qquad \omega_2 = x_2(t_0), \qquad (6.9)$$

$$w_1(t, \boldsymbol{x}, u) = x_2, \qquad w_2(t, \boldsymbol{x}, u) = u - 2(x_1 + x_2). \qquad (6.10)$$

Equations (6.2) and (6.5)–(6.7) follow from (6.1), equations (6.3) and (6.10) from (4.6), and equations (6.4), (6.8) and (6.9) are initial conditions (4.7) rewritten in the new notation. System (6.5)–(6.7) is the classical problem of optimal control with $u(t)$ being the control parameter. We determine $u(t)$, $t_0 \leqslant t \leqslant t_1$ to give the minimal value of the functional $\mathfrak{R}_0(\boldsymbol{x}, u)$ under the additional restrictions (6.3) and (6.4). It is important to notice that we do not require that $0 \leqslant u(t) \leqslant 1$, though these inequalities are satisfied in the examples considered below.

In order to solve (6.2)–(6.4), we introduce new unknown functions $p_1(t)$ and $p_2(t)$ and construct the Lagrange function

$$L(t, \boldsymbol{x}, u, \boldsymbol{p}) = \sum_{i=0}^{2} \lambda_i h_i(t, \boldsymbol{x}, u) + \boldsymbol{p}(t)[\dot{\boldsymbol{x}} - \boldsymbol{w}(t, \boldsymbol{x}, u)]$$

and the function

$$l(t_0, \boldsymbol{x}(t_0), t_1, \boldsymbol{x}(t_1)) = \sum_{i=0}^{2} \lambda_i \omega_i(t_0, \boldsymbol{x}(t_0), t_1, \boldsymbol{x}(t_1)),$$

where $\lambda_0$, $\lambda_1$, $\lambda_2$ are the Lagrange multipliers. The necessary conditions for the parameter $u(t)$ to be optimal in a weak sense are (Pontryagin *et al.* 1962)

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right) + \frac{\partial L}{\partial \boldsymbol{x}} = 0, \tag{6.11}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{x}}}(t_k) = (-1)^k \frac{\partial l}{\partial \boldsymbol{x}(t_k)} \quad (k = 0, 1), \tag{6.12}$$

$$\frac{\partial L}{\partial u} = 0, \tag{6.13}$$

$$\lambda_0 \geqslant 0. \tag{6.14}$$

If $\lambda_0 = 0$, then equations (6.11)–(6.13), (6.3) and (6.4) give $\lambda_1 = 0$ and $\lambda_2 = 0$, which means that the function $u(t)$ cannot be found. Therefore $\lambda_0 > 0$ and it can be taken as any positive constant. It is convenient to take $\lambda_0 = \frac{1}{2}$. In this case equations (6.3), (6.4) and (6.11)–(6.13) lead to the following boundary-value problem:

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= p_2 - x_1 - \tfrac{3}{2}x_2, \\ \dot{p}_1 &= p_2 - x_1 - \tfrac{1}{2}x_2, \\ \dot{p}_2 &= \tfrac{3}{2}p_2 - p_1 - \tfrac{1}{2}x_1 - \tfrac{1}{4}x_2, \end{aligned}\right\} \tag{6.15}$$

$$x_1(t_0) = 0, \quad x_2(t_0) = 0, \quad p_1(t_1) = \tfrac{1}{2}, \quad p_2(t_1) = \tfrac{1}{2}. \tag{6.16}$$

Once (6.15) and (6.16) have been solved, the optimal function $u(t)$ is given by

$$u(t) = p_2 + x_1 + \tfrac{1}{2}x_2. \tag{6.17}$$

We cannot prove here that the body velocity $u(t)$ determined by equations (6.15)–(6.17) gives the minimal value of the energy $E_{ac}$. We may only conclude that if the optimal velocity of the body, which minimizes the acoustic energy $E_{ac}$, exists, then it has to satisfy equations (6.15)–(6.17).

The linear boundary-value problem (6.15) and (6.16) is solved numerically. The acoustic energy $E_{ac}$ for the optimal acceleration of the body is shown in figure 8 as a function of the duration of the acceleration stage $1/\alpha$. For comparison the acoustic energy for constant acceleration of the body, $f(\tau) = \tau$, $0 < \tau < 1$, is shown with a thin line. In this case the acoustic energy $E_{ac}(1/\alpha)$ is given analytically (Longhorn 1952)

$$E_{ac}(1/\alpha) = \tfrac{1}{2}\alpha^2\{1 - \mathrm{e}^{-1/\alpha}[\sin(1/\alpha) + \cos(1/\alpha)]\}.$$

The optimal sphere acceleration, and the evolution of both the pressure $p(1, 0, t)$ and the work $A(t)$ with time are shown in figures 9 and 10 for durations of the

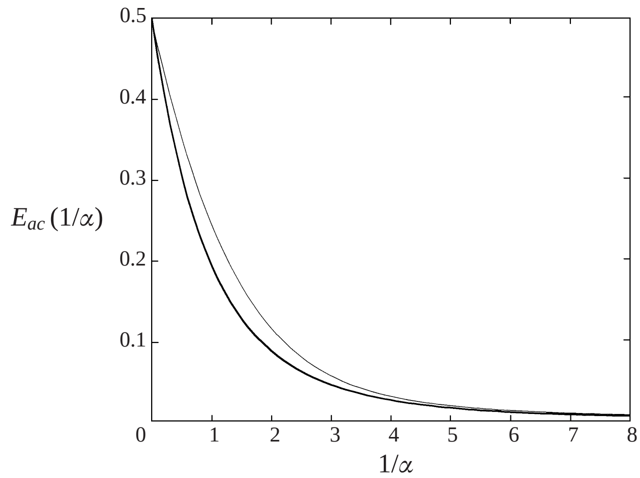


FIGURE 8. Acoustic energy as a function of the duration of the acceleration, $1/\alpha$. The thick line is for the optimal acceleration, and the thin line is for uniform acceleration.

acceleration stage of 1 and 10. It is seen that the optimal way to accelerate the sphere up to a given velocity is, roughly speaking, a combination of impulsive motions of the body at the beginning and end of the acceleration stage with uniform acceleration in between.

## 7. Free motion initiated by an initial impulse

Up to this point forced body motion with a prescribed velocity $Vf(t'/T_a)$ has been considered. A similar analysis can be applied to the problem of the free motion of a half-submerged sphere subject to an initial impulse. That is: a half-submerged sphere of mass $M$ at rest suddenly starts to move down with an initial velocity $V$. Now the duration of the acceleration stage $T_a$ is not defined, in contrast to the forced motion case. We assume that $T_a = O(R/c_0)$ and put $\alpha = 1$ in the following analysis. The body velocity $Vf(c_0t'/R)$ subsequently changes due to both the inertia of the body and the action of the induced hydrodynamic pressure, and has to be determined together with the liquid flow and the pressure distribution. The hydrodynamic force on the hemispherical surface equals half that on a fully submerged sphere: thus the problem of free motion after an impulse is the same as that considered by Taylor (1942).

The non-dimensional variables used below are the same as in the forced motion problem (see § 3) except that the liquid velocity is now scaled with the initial body velocity, not its final velocity. Body motion is governed by Newton's law which in dimensionless variables has the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{3}{\pi}\frac{m_a}{M}F(t) \quad (t > 0), \quad f(0) = 1, \tag{7.1}$$

where the hydrodynamic force on the hemispherical surface $F(t)$ is given by (4.17) and $m_a$ is the added mass of the half-submerged sphere, $\frac{1}{3}\pi\rho R^3$. Substituting (4.17) into (7.1) and integrating the result, we obtain the equation

$$f(t) - 1 = 2\frac{m_a}{M}\phi(1, t). \tag{7.2}$$

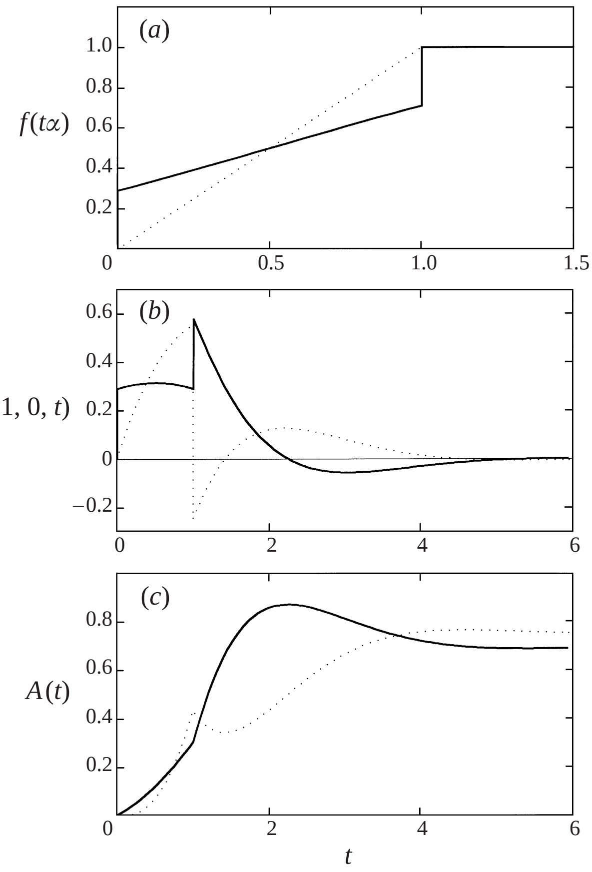


FIGURE 9. Optimal acceleration of the sphere for the dimensionless duration of the acceleration stage equal to unity: (*a*) the velocity of the sphere, (*b*) pressure $p(1,0,t)$ at the base of the sphere, (*c*) the work done, $A(t)$. The solid lines show the case of optimal acceleration and the dotted lines the case of uniform acceleration.

Using (4.5) and (4.6) gives an equation for $\psi(\xi)$:

$$\psi_{\xi\xi} + 2a\psi_{\xi} + 2a\psi = 1 \quad (\xi > 0), \qquad (7.3)$$
$$\psi(0) = \psi_{\xi}(0) = 0,$$

where $a = 1 + m_a/M$. This is readily solved: for example in the case where $a < 2$, and defining $b = \sqrt{1-(m_a/M)^2}$ the resulting motion and base pressure are

$$f(t) = u_r(1,0,t) = \frac{1}{a} + \frac{a-1}{a}e^{-at}\cos bt - \frac{a-1}{b}e^{-at}\sin bt, \qquad (7.4)$$

$$p(1,0,t) = e^{-at}\left[\cos bt - \frac{a-1}{b}\sin bt\right]. \qquad (7.5)$$

The dimensional force on the object is $\frac{2}{3}\pi\rho_0 V c_0 R^2 p(1,0,t)$, which does change sign as in the forced motion case, and oscillates for $a < 2$. For the case $a > 2$ the trigonometric functions are simply replaced by the corresponding hyperbolic functions, and $a = 2$ is the case of critical damping. The velocity of the body, (7.4), does not change sign and exponentially approaches the limiting value $1/a$ as $\xi \to \infty$ for any $a > 1$.

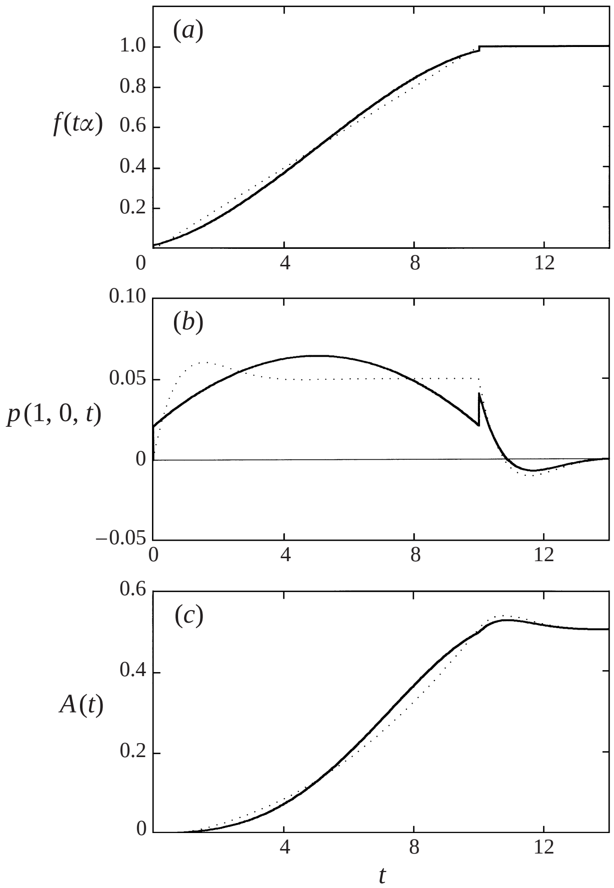


FIGURE 10. As figure 9 but for the dimensionless duration of the acceleration stage equal to 10.

In dimensional terms this velocity equals $VM/(M+m_a)$, the value obtained from the pressure-impulse theory. The case of the free floating half-submerged sphere, $M=2m_a$, gives $a=3/2$. Heavy spheres, $1<a<3/2$, and light spheres, $a>3/2$, are assumed to be kept half-submerged before the impact by external forces.

The body energy before the impact, $E'_B(0)=MV^2/2$, and its energy after the impact, $E'_B(\infty)=MV^2/(2a^2)$, are in dimensionless variables $E_B(0)=1/[2(a-1)]$ and $E_B(\infty)=1/[2a^2(a-1)]$, respectively. The total energy of the liquid after the whole impact event is $A(\infty)=(a+1)/(2a^2)$, consistent with conservation of energy. The manipulations leading to equations (5.10) and (5.11) show that the total kinetic energy of the liquid $T(\infty)=(a+2)/(4a^2)$ and the potential energy $\Pi(\infty)=1/(4a)$ at the end of the acoustic stage. It is seen that the energy conservation law $E_B(0)=E_B(\infty)+A(\infty)$ is satisfied.

The kinetic energy of the liquid flow after the impact predicted by the pressure-impulse theory is $\hat{T}=1/(2a^2)$ and corresponds to the incompressible flow close to the moving body. Thus, the dimensionless acoustic energy $E_{ac}=1/(2a)$. The fraction of the acoustic energy

$$\frac{E_{ac}}{A(\infty)}=\frac{a}{a+1}=\frac{M+m_a}{2M+m_a}$$

is greater than $\frac{1}{2}$, which implies that acoustic waves radiate more than 50% of the total liquid energy for the case of free motion, approaching 100% as the body mass $M \to 0$ when the body has negligible velocity at the end of the impact. Note that this fraction is independent of Mach number.

## 8. Conclusion

It is shown that for a downward impact on a floating half-submerged sphere pressure-impulse theory gives the liquid flow and the free-surface elevation close to the body after the acceleration stage. The energy loss predicted by this theory is the acoustic energy carried away with the compression wave, which is generated by the impulsive start of the sphere. If the acceleration stage is not of zero duration, the acoustic energy decreases and can be minimized, for given acceleration duration, by an optimal choice of the body motion. The flow close to the body tends exponentially to that given by the pressure-impulse theory and does not depend to leading order on details of the body acceleration. The acoustic energy is localized in a small neighbourhood of the outward-propagating compression wave. The flow in this neighbourhood is approximately one-dimensional with equipartition of the acoustic kinetic and potential energies. The acoustic region of the flow is separated from the main flow by an intermediate region, where the liquid is almost at rest. The kinetic energy of the main flow is as given by the pressure-impulse approach. Both the pressure on the body surface and the hydrodynamic force oscillate with time as they decay, the magnitude of the negative pressures being quite small. The work done to oppose the hydrodynamic force is not a monotonic function of time. The work peaks at the moment when the pressure at the sphere's base drops to atmospheric pressure. If the velocity does not increase monotonically but has a maximum value there seems to be no bound on the fraction of energy radiated in the acoustic wave.

It is reasonable to suppose that this type of behaviour also occurs for impulsive motion of smooth bodies of other shapes which also have a vertical tangent to the surface at the waterline. If some of the body surface overhangs at the waterline it would cause jet formation and the energy contained in these jets would affect the overall energy distribution. It is less clear what happens for in the case of a surface that recedes at the waterline, like a three-quarters submerged sphere. We note that in this case classical experiments (Worthington 1908) with spheres entering a liquid show that the flow may or may not converge over the sphere's surface, depending on its roughness and velocity.

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